

SOME NOTES ON THE INCLUSION BETWEEN MORREY SPACES

by Philotheus Tuerah

Submission date: 06-May-2022 10:16AM (UTC+0700)

Submission ID: 1829525877

File name: JURNAL_TUERAH_3.pdf (161.82K)

Word count: 2810

Character count: 10613

SOME NOTES ON THE INCLUSION BETWEEN MORREY SPACES

PHILOTHEUS E. A. TUERAH AND NICKY K. TUMALUN*

(Communicated by L. Liu)

6

Abstract. In this paper, we show that the Morrey spaces $\mathcal{M}_{q_1}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ for $q_1 \neq q_2$. We also show that the vanishing Morrey spaces $\mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n)$ are not empty and properly contained in the Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$.

1. Introduction

4

Let $1 \leq p \leq q < \infty$ and $n \geq 2$. The Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is the set of all functions $f \in L_{\text{loc}}^p(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}_q^p} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{\frac{1}{q} - \frac{1}{p}} \|f\|_{L^p(B(x, r))} < \infty,$$

where

$$\|f\|_{L^p(B(x, r))} = \left(\int_{B(x, r)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Here $B(x, r)$ is the open ball in Euclidean space \mathbb{R}^n with center x and radius r , and $|B(x, r)|$ denotes its Lebesgue measure. Meanwhile, the weak Morrey space $w\mathcal{M}_q^p(\mathbb{R}^n)$ is defined to be the set of all functions $f \in wL_{\text{loc}}^p(\mathbb{R}^n)$ for which

$$\|f\|_{w\mathcal{M}_q^p} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{\frac{1}{q} - \frac{1}{p}} \|f\|_{wL^p(B(x, r))} < \infty,$$

where

$$\|f\|_{wL^p(B(x, r))} = \sup_{t > 0} t |\{y \in B(x, r) : |f(y)| > t\}|^{\frac{1}{p}},$$

and $|\{y \in B(x, r) : |f(y)| > t\}|$ also denotes the Lebesgue measure of the set $\{y \in B(x, r) : |f(y)| > t\}$. Now, we define

$$\mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n) = \left\{ f \in \mathcal{M}_q^p(\mathbb{R}^n) : \lim_{r \rightarrow 0} \mathcal{M}_f(r) = 0 \right\},$$

Mathematics subject classification (2020): 46E30, 42B35.

Keywords and phrases: Morrey spaces, vanishing Morrey spaces.

* Corresponding author.

where

$$\mathcal{M}_f(r) = \sup_{x \in \mathbb{R}^n} |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(B(x,r))}.$$

The set $\mathcal{VM}_q^p(\mathbb{R}^n)$ is called the *vanishing Morrey space*. It is clear that $\mathcal{VM}_q^p(\mathbb{R}^n)$ is a subset of $\mathcal{M}_q^p(\mathbb{R}^n)$.

The Morrey spaces were introduced by C. B. Morrey [1] and the vanishing Morrey spaces were introduced in [2]. Recently, many authors are attracted in studying the inclusion properties between Morrey spaces [3, 4, 5, 6, 7, 8]. One interesting result stated in [5, Remark 4.5], that is, the weak Morrey spaces $w\mathcal{M}_{q_1}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ and vice versa, for distinct values q_1 and q_2 . This statement was deduced by a characterization of inclusion between weak Morrey spaces and its parameters, which is proved by using Closed Graph Theorem and Morrey norm estimate for the characteristic functions of balls [5, Theorem 4.4]. Regarding to the inclusion between vanishing Morrey spaces and Morrey spaces over a bounded domain, it was stated in [9] that the vanishing Morrey spaces are properly contained in the Morrey spaces without giving an explicit counter example.

In this paper, we will prove that the Morrey spaces $\mathcal{M}_{q_1}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ and the Morrey spaces $\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathcal{M}_{q_1}^p(\mathbb{R}^n)$, for different values q_1 and q_2 . This result is more general and sharp than the previous result in [5] since we can recover that previous result and the fact that $\mathcal{M}_q^p(\mathbb{R}^n)$ is a proper subset of $w\mathcal{M}_q^p(\mathbb{R}^n)$ [6, Theorem 1.2]. We also note that our method here is different than in [5] because we give a function which belongs to $\mathcal{M}_{q_1}^p(\mathbb{R}^n) \setminus w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ and a function which belongs to $\mathcal{M}_{q_2}^p(\mathbb{R}^n) \setminus w\mathcal{M}_{q_1}^p(\mathbb{R}^n)$. Furthermore, by using the idea in [8], we also show that the vanishing Morrey spaces $\mathcal{VM}_q^p(\mathbb{R}^n)$ are non empty and properly contained in $\mathcal{M}_q^p(\mathbb{R}^n)$ by providing some examples.

The positive constant C that appears in the proofs of all theorems may vary from line to line and the notation $C = C(n, p, q)$ indicates that C depends only on n, p and q .

2. A note on the inclusion between weak Morrey spaces

Let $1 \leq p < q < \infty$ and $\gamma = \frac{n}{q} < n$. Define a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by the formula

$$f(y) = \begin{cases} |y|^{-\gamma}, & y \neq 0, \\ 0, & y = 0. \end{cases} \tag{1}$$

It is clear that $1 - \frac{n}{p\gamma} < 0$ and $n - p\gamma > 0$ by observing to the given assumptions.

The function f , that appears in Lemma 1 and 3, is defined by (1).

LEMMA 1. If $x \in \mathbb{R}^n$ and $r > 0$, then $\|f\|_{L^p(B(x,r))} \leq Cr^{\frac{n}{p}-\gamma} = Cr^{\frac{n}{p}-\frac{n}{q}}$, where $C = C(n, p, q)$.

Proof. Note that

$$\int_{B(x,r)} |f(y)|^p dy = \int_{\{|y| \leq |x-y| < r\}} |y|^{-p\gamma} dy + \int_{\{|x-y| < |y|\} \cap \{|x-y| < r\}} |y|^{-p\gamma} dy = I + II.$$

Since $n - p\gamma > 0$, we have

$$I \leq \int_{\{|y| < r\}} |y|^{-p\gamma} dy = C \int_0^r t^{n-p\gamma-1} dt = Cr^{n-p\gamma}$$

and

$$II \leq \int_{\{|x-y| < r\}} |x-y|^{-p\gamma} dy = C \int_0^r t^{n-p\gamma-1} dt = Cr^{n-p\gamma},$$

by using polar coordinate for radial function. Therefore

$$\|f\|_{L^p(B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} \leq C \left(Cr^{\frac{n}{p}-\gamma} \right)^{\frac{1}{p}} = Cr^{\frac{n}{p}-\frac{n}{q}},$$

which proves the lemma. \square

The following lemma is not hard to prove. We leave its proof to the reader.

LEMMA 2. Let $s > 0$, $M \geq 0$, and $\varphi : (0, \infty) \rightarrow [0, \infty)$. If

$$\sup_{0 < t \leq s} \varphi(t) = M = \sup_{s < t < \infty} \varphi(t),$$

then

$$\sup_{t > 0} \varphi(t) = M.$$

Using the above lemma, we can compute the weak Lebesgue norm of f on the ball $B(0, r)$ with arbitrary radius r .

LEMMA 3. If $r > 0$, then $\|f\|_{wL^p(B(0,r))} = Cr^{-\gamma+\frac{n}{p}} = Cr^{-\frac{n}{q}+\frac{n}{p}}$, where $C = C(n, p, q)$.

Proof. Let r be an arbitrary positive real number. Note that, for every $t > 0$, we have

$$\begin{aligned} |\{y \in B(0, r) : |f(y)| > t\}| &= \left| \left\{ y \in B(0, r) : |y| < t^{-\frac{1}{\gamma}} \right\} \right| \\ &= \left| B(0, r) \cap B(0, t^{-\frac{1}{\gamma}}) \right|. \end{aligned} \tag{2}$$

We now define $\varphi : (0, \infty) \rightarrow [0, \infty)$ by the formula

$$\varphi(t) = t |\{y \in B(0, r) : |f(y)| > t\}|^{\frac{1}{p}}. \tag{3}$$

For every $t > r^{-\gamma}$, we obtain $t^{-\frac{1}{\gamma}} < r$. Then

$$|\{y \in B(0, r) : |f(y)| > t\}| = |B(0, t^{-\frac{1}{\gamma}})| = Ct^{-\frac{n}{\gamma}},$$

by using (2). This gives us

$$t|\{y \in B(0, r) : |f(y)| > t\}|^{\frac{1}{p}} = Ct\left(t^{-\frac{n}{\gamma}}\right)^{\frac{1}{p}} = Ct^{1-\frac{n}{p\gamma}}, \quad \forall t \in (r^{-\gamma}, \infty). \tag{4}$$

On the other hand, for every $t \leq r^{-\gamma}$, we have $t^{-\frac{1}{\gamma}} \geq r$. Hence

$$|\{y \in B(0, r) : |f(y)| > t\}| = |B(0, r)| = Cr^n,$$

which comes from (2). Therefore,

$$t|\{y \in B(0, r) : |f(y)| > t\}|^{\frac{1}{p}} = Ct(r^n)^{\frac{1}{p}} = Ctr^{\frac{n}{p}}, \quad \forall t \in (0, r^{-\gamma}]. \tag{5}$$

We obtain

$$\varphi(t) = \begin{cases} Ct^{1-\frac{n}{p\gamma}}, & \forall t \in (r^{-\gamma}, \infty) \\ Ctr^{\frac{n}{p}}, & \forall t \in (0, r^{-\gamma}]. \end{cases}$$

by virtue of (4) and (5). Observing φ non increasing on $(r^{-\gamma}, \infty)$ and non decreasing on $(0, r^{-\gamma}]$, since $1 - \frac{n}{p\gamma} < 0$ and $\frac{n}{p} > 0$ respectively, then

$$\sup_{r^{-\gamma} < t < \infty} \varphi(t) = Cr^{-\gamma+\frac{n}{p}} = \sup_{0 < t \leq r^{-\gamma}} \varphi(t). \tag{6}$$

Thus

$$\|f\|_{wL^p(B(0,r))} = \sup_{t>0} \varphi(t) = Cr^{-\gamma+\frac{n}{p}} = Cr^{-\frac{n}{q}+\frac{n}{p}},$$

that is concluded from Lemma 2. \square

16

By taking Lemma 2 and Lemma 3 as the tools, we are ready to state and prove the first main result of this paper.

THEOREM 1. *Let $1 \leq p < q_1 < \infty$ and $1 \leq p < q_2 < \infty$. If $q_1 \neq q_2$, then $\mathcal{M}_{q_1}^p(\mathbb{R}^n) \not\subseteq w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ and $\mathcal{M}_{q_2}^p(\mathbb{R}^n) \not\subseteq w\mathcal{M}_{q_1}^p(\mathbb{R}^n)$.*

Proof. We will only prove that $\mathcal{M}_{q_1}^p(\mathbb{R}^n)$ is not contained by $w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$. The proof that $\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ is not contained by $w\mathcal{M}_{q_1}^p(\mathbb{R}^n)$ can be done by similar method.

Let $\gamma_1 = n/q_1$ and $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by the formula

$$f_1(y) = \begin{cases} |y|^{-\gamma_1}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

We will show that $f_1 \in \mathcal{M}_{q_1}^p(\mathbb{R}^n) \setminus w\mathcal{M}_{q_2}^p(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$ and $r > 0$ be arbitrarily given. According to Lemma 1, by replacing γ with γ_1 , we obtain

$$|B(x, r)|^{\frac{1}{q_1} - \frac{1}{p}} \|f_1\|_{L^p(B(x,r))} \leq Cr^{\frac{n}{q_1} - \frac{n}{p} - \frac{n}{q_1} - \frac{n}{q_1}} = C < \infty.$$

This gives us

$$\|f_1\|_{\mathcal{M}_{q_1}^p} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{\frac{1}{q_1} - \frac{1}{p}} \|f_1\|_{L^p(B(x, r))} < \infty,$$

since x and r are arbitrary. Whence $f_1 \in \mathcal{M}_{q_1}^p(\mathbb{R}^n)$. By virtue to Lemma 3, we have

$$\|f_1\|_{w.\mathcal{M}_{q_2}^p} \geq |B(0, r)|^{\frac{1}{q_2} - \frac{1}{p}} \|f_1\|_{wL^p(B(0, r))} = Cr^{\frac{n}{q_2} - \frac{n}{p}} r^{\frac{n}{p} - \frac{n}{q_1}} = Cr^n \left(\frac{1}{q_2} - \frac{1}{q_1}\right).$$

Hence $\|f_1\|_{w.\mathcal{M}_{q_2}^p} = \infty$. This is due to arbitrary r and $q_1 \neq q_2$. We conclude that $f_1 \notin w.\mathcal{M}_{q_2}^p(\mathbb{R}^n)$. Thus, we have already proved that $\mathcal{M}_{q_1}^p(\mathbb{R}^n) \not\subseteq w.\mathcal{M}_{q_2}^p(\mathbb{R}^n)$. \square

As an immediate consequence of Theorem 1, we recover the result from [5] which is stated in the following corollary.

COROLLARY 1. *Let $1 \leq p < q_1 < \infty$ and $1 \leq p < q_2 < \infty$. If $q_1 \neq q_2$, then $w.\mathcal{M}_{q_1}^p(\mathbb{R}^n) \not\subseteq w.\mathcal{M}_{q_2}^p(\mathbb{R}^n)$ and $w.\mathcal{M}_{q_2}^p(\mathbb{R}^n) \not\subseteq w.\mathcal{M}_{q_1}^p(\mathbb{R}^n)$.*

3. A note on the inclusion between Morrey spaces and vanishing Morrey spaces

Let $1 \leq p < q < \infty$ and $\delta = \exp\left(\frac{-2q}{np}\right)$. Define a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by the formula

$$g(y) = \begin{cases} \left(\frac{\chi_B(y)}{|y|^{\frac{np}{q}} (\ln|y|)^2}\right)^{\frac{1}{p}}, & y \neq 0, \\ 0, & y = 0, \end{cases} \tag{7}$$

where χ_B is a characteristic function defined on $B = B(0, \delta)$.

The function g in the following lemma is defined by (7). This following lemma shows that the vanishing Morrey spaces in a non empty set.

LEMMA 4. $g \in \mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$ be arbitrarily given. Note that

$$\begin{aligned} |B(x, r)|^{\frac{1}{q} - \frac{1}{p}} \|g\|_{L^p(B(x, r))} &\leq C \left(\int_{|y| \leq |x|} \frac{\chi_B(y)}{|x-y|^{n-\frac{np}{q}} |y|^{\frac{np}{q}} (\ln|y|)^2} dy \right)^{\frac{1}{p}} \\ &\quad + C \left(\int_{\substack{\{|x-y| < |y|\} \\ \cap \{|x-y| < r\}}} \frac{\chi_B(y)}{|x-y|^{n-\frac{np}{q}} |y|^{\frac{np}{q}} (\ln|y|)^2} dy \right)^{\frac{1}{p}} \\ &= I + II. \end{aligned} \tag{8}$$

Now we have two cases, that is, $\delta \leq r$ or $r < \delta$. Assume $\delta \leq r$, then we have

$$I \leq \int_{|y| < r} \frac{\chi_B(y)}{|y|^n (\ln|y|)^2} dy = \int_{|y| < \delta} \frac{1}{|y|^n (\ln|y|)^2} dy = C \left(\frac{-1}{\ln(\delta)}\right), \tag{9}$$

and

$$\begin{aligned}
 II &= \int_{\substack{\{|x-y| < |y| < \delta\} \\ \cap \{|x-y| < r\}}} \frac{1}{|x-y|^{n-\frac{np}{q}} |y|^{\frac{np}{q}} (\ln |y|)^2} dy \leq \int_{|x-y| < \delta} \frac{1}{|x-y|^n (\ln |x-y|)^2} dy \\
 &= C \left(\frac{-1}{\ln(\delta)} \right), \tag{10}
 \end{aligned}$$

since $1/t^{np/q}(\ln(t))^2$ decreasing on interval $(0, \delta)$. Assume $r < \delta$. We have

$$I \leq \int_{|y| < r} \frac{1}{|y|^n (\ln |y|)^2} dy = C \left(\frac{-1}{\ln(r)} \right) \leq C \left(\frac{-1}{\ln(\delta)} \right), \tag{11}$$

and

$$\begin{aligned}
 II &= \int_{\substack{\{|x-y| < |y| < r\} \\ \cap \{|x-y| < r\}}} \frac{1}{|x-y|^{n-\frac{np}{q}} |y|^{\frac{np}{q}} (\ln |y|)^2} dy \leq \int_{|x-y| < r} \frac{1}{|x-y|^n (\ln |x-y|)^2} dy \\
 &= C \left(\frac{-1}{\ln(r)} \right) \leq C \left(\frac{-1}{\ln(\delta)} \right), \tag{12}
 \end{aligned}$$

since $1/t^{np/q}(\ln(t))^2$ decreasing on interval $(0, r) \subseteq (0, \delta)$. By virtue of (8), (9), (10), (11), and (12), we conclude that

$$|B(x, r)|^{\frac{1}{q} - \frac{1}{p}} \|g\|_{L^p(B(x, r))} \leq I + II \leq C \left(\frac{-1}{\ln(\delta)} \right)^{\frac{1}{p}},$$

where $C = C(n, p, q)$. This means $g \in \mathcal{M}_q^p(\mathbb{R}^n)$. We remaind to prove

$$\lim_{r \rightarrow 0} \mathcal{M}_f(r) = 0. \tag{13}$$

For every $0 < r < \delta$, we have shown that

$$\mathcal{M}_f(r) \leq C \left(\frac{-1}{\ln(r)} \right)^{\frac{1}{p}}.$$

This means (13) holds and the proof is done. \square

Now we define a function that will play as an element of Morrey spaces but not in the vanishing Morrey spaces. Let $1 \leq p < q < \infty$. For every $k \in \mathbb{N}$, with $k \geq 3$, we set $x_k = (2^{-k}, \dots, 0) \in \mathbb{R}^n$ and

$$u_k(y) = \begin{cases} 8^{\frac{np}{q}k}, & y \in B(x_k, 8^{-k}), \\ 0, & y \notin B(x_k, 8^{-k}). \end{cases}$$

Define a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ by the formula

$$u(y) = \left(\sum_{k=3}^{\infty} u_k(y) \right)^{\frac{1}{p}}. \tag{14}$$

We first claim that u belongs to the Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$.

LEMMA 5. $u \in \mathcal{M}_q^p(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$ be arbitrarily given. There are two cases: (i) $x \notin B(x_k, 2(4^{-k}))$ for every $k \geq 3$, or, (ii) $x \in B(x_j, 2(4^{-j}))$ for some $j \geq 3$. Assume (i) holds. Then

$$2(4^{-k}) \leq |x - x_k| \leq |x - y| + |y - x_k| < r + 4^{-k},$$

for every $y \in B(x, r) \cap B(x_k, 8^{-k})$. This means $r^{\frac{np}{q}-n} \leq 4^{(n-\frac{np}{q})k}$ and

$$\begin{aligned} r^{\frac{np}{q}-n} \int_{B(x,r)} |u(y)|^p dy &\leq \sum_{k=3}^{\infty} 4^{(n-\frac{np}{q})k} \int_{B(x,r) \cap B(x_k, 8^{-k})} 8^{\frac{np}{q}k} dy \\ &\leq C \sum_{k=3}^{\infty} 2^{(\frac{np}{q}-n)k} < \infty, \end{aligned} \tag{15}$$

where C depends on n . Assume (ii) holds. Since $\{B(x_k, 2(4^{-k}))\}_{k \geq 3}$ is a disjoint collection, then there is only one $j \geq 3$ such that $x \in B(x_j, 2(4^{-j}))$ and $x \notin B(x_k, 2(4^{-k}))$ for every $k \geq 3$ with $k \neq j$. Note that

$$r^{\frac{np}{q}-n} \int_{B(x,r) \cap B(x_j, 8^{-j})} u_j(y) dy = r^{\frac{np}{q}-n} \int_{B(x,r) \cap B(x_j, 8^{-j})} 8^{\frac{np}{q}j} dy \leq C < \infty, \tag{16}$$

where C depends on n, p , and q . By virtue of (16) and the computation of (15), we have

$$\begin{aligned} r^{\frac{np}{q}-n} \int_{B(x,r)} |u(y)|^p dy &= r^{\frac{np}{q}-n} \sum_{k=3}^{\infty} \int_{B(x,r) \cap B(x_k, 8^{-k})} u_k(y) dy \\ &= r^{\frac{np}{q}-n} \int_{B(x,r) \cap B(x_j, 8^{-j})} u_j(y) dy \\ &\quad + r^{\frac{np}{q}-n} \sum_{\substack{k=3 \\ k \neq j}}^{\infty} \int_{B(x,r) \cap B(x_k, 8^{-k})} u_k(y) dy \\ &\leq C + C \sum_{\substack{k=3 \\ k \neq j}}^{\infty} 2^{(\frac{np}{q}-n)k} < \infty, \end{aligned} \tag{17}$$

where C depends on n, p , and q . Combining (16) and (17), whence

$$|B(x, r)|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(B(x,r))} = C \left(r^{\frac{np}{q}-n} \int_{B(x,r)} |u(y)|^p dy \right)^{\frac{1}{p}} \leq C < \infty,$$

where C depends on n, p , and q . Therefore $u \in \mathcal{M}_q^p(\mathbb{R}^n)$. \square

The following theorem states that the vanishing Morrey spaces is a non empty proper subset of the Morrey spaces. This theorem is the second main result in this paper.

THEOREM 2. *Let $1 \leq p < q < \infty$. Then $\mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n)$ is a non empty proper subset of $\mathcal{M}_q^p(\mathbb{R}^n)$.*

Proof. According to Lemma 4, $\mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n)$ is non empty, and according to Lemma 5, the function u belongs to $\mathcal{M}_q^p(\mathbb{R}^n)$. Therefore, we need only to show that u does not belong to $\mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n)$. Let $0 < r < 1$. By the Archimedean property, there is an integer $k \geq 3$ such that $8^{-k} < r$. Then

$$\begin{aligned} (\mathcal{M}_f(r))^p &\geq Cr^{\frac{np}{q}-n} \int_{B(x_k, r)} |u(y)|^p dy \geq C \int_{B(x_k, 8^{-k})} u_k(y) dy \\ &= C \int_{B(x_k, 8^{-k})} 8^{\frac{np}{q}k} dy \geq C8^{-nk} \int_{B(x_k, 8^{-k})} 1 dy = C > 0, \end{aligned}$$

where C depends on n . This means $\mathcal{M}_f(r)$ is bounded away from zero as r tends to zero. Thus $u \notin \mathcal{V}\mathcal{M}_q^p(\mathbb{R}^n)$. \square

REFERENCES

- [1] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, (1938), 126–166.
- [2] C. VITANZA, *Functions with vanishing Morrey norm and elliptic partial differential equations*, in Proceedings of Methods of Real Analysis and Partial Differential Equations, Capri, (1990).
- [3] D. I. HAKIM AND Y. SAWANO, *Complex interpolation of various subspaces of Morrey spaces*, Sci. China Math., **63**, 5 (2020), 937–964.
- [4] H. GUNAWAN, D. I. HAKIM, AND M. IDRIS, *Proper inclusions of Morrey spaces*, Glasnik Matematički., **53**, 1 (2018), 143–151.
- [5] H. GUNAWAN, D. I. HAKIM, K. M. LIMANTA, AND A. A. MASTA, *Inclusion properties of generalized Morrey spaces*, Math. Nachr., **290**, 1 (2017), 332–340.
- [6] H. GUNAWAN, D. I. HAKIM, E. NAKAI, AND Y. SAWANO, *On inclusion relation between weak Morrey spaces and Morrey spaces*, Nonlinear Analysis., **168**, (2018), 27–31.
- [7] N. K. TUMALUN AND H. GUNAWAN, *Morrey spaces are embedded between weak Morrey spaces and Stummel classes*, J. Indones. Math. Soc., **253**, (2019), 203–209.
- [8] N. K. TUMALUN, D. I. HAKIM, AND H. GUNAWAN, *Inclusion between generalized Stummel classes and other function spaces*, Math. Inequal. Appl., **232**, (2020), 547–562.
- [9] N. SAMKO, *Maximal, potential and singular operators in vanishing generalized Morrey spaces*, J. Global. Optim., **57**, (2013), 1385–1399.

(Received March 22, 2021)

Philotheus E. A. Tuerah
Mathematics Department, Faculty of Mathematics and Natural Sciences
Universitas Negeri Manado, Tondano Selatan District, Minahasa Regency
Sulawesi Utara Province, Indonesia
e-mail: pheaturah@unima.ac.id

Nicky K. Tumalun
Mathematics Department, Faculty of Mathematics and Natural Sciences
Universitas Negeri Manado, Tondano Selatan District, Minahasa Regency
Sulawesi Utara Province, Indonesia
e-mail: nickytumalun@unima.ac.id

SOME NOTES ON THE INCLUSION BETWEEN MORREY SPACES

ORIGINALITY REPORT

13%

SIMILARITY INDEX

8%

INTERNET SOURCES

10%

PUBLICATIONS

%

STUDENT PAPERS

PRIMARY SOURCES

- | | | |
|---|--|-----|
| 1 | Piotr Bogusław Mucha. "Zygmund Spaces, Inviscid Limit and Uniqueness of Euler Flows", Communications in Mathematical Physics, 06/2008
Publication | 1% |
| 2 | doi.org
Internet Source | 1% |
| 3 | Denny Ivanal Hakim, Yoshihiro Sawano. "Complex interpolation of various subspaces of Morrey spaces", Science China Mathematics, 2020
Publication | 1% |
| 4 | ems.press
Internet Source | 1% |
| 5 | cvgmt.sns.it
Internet Source | 1% |
| 6 | jims-a.org
Internet Source | 1% |
| 7 | Lars Diening. "Capacities", Lecture Notes in Mathematics, 2011
Publication | 1% |
| 8 | Denny Ivanal Hakim, Eiichi Nakai, Yoshihiro Sawano. "Generalized fractional maximal operators and vector-valued inequalities on generalized Orlicz–Morrey spaces", Revista Matemática Complutense, 2015
Publication | <1% |
-

9	Dongli Liu, Jiman Zhao. "Multilinear Hausdorff operators on weighted Herz and Morrey–Herz spaces with variable exponent", Journal of Pseudo-Differential Operators and Applications, 2022 Publication	<1 %
10	real-j.mtak.hu Internet Source	<1 %
11	Guy Barles. "Nonlocal First-Order Hamilton-Jacobi Equations Modelling Dislocations Dynamics", Communications in Partial Differential Equations, 8/1/2006 Publication	<1 %
12	theicmme.org Internet Source	<1 %
13	A. V. Pokrovskii. "Removable singularities of solutions of second-order divergence-form elliptic equations", Mathematical Notes, 03/2005 Publication	<1 %
14	Humberto Rafeiro, Stefan Samko. "BMO–VMO results for fractional integrals in variable exponent Morrey spaces", Nonlinear Analysis, 2019 Publication	<1 %
15	Jost. "Strong Solutions", Graduate Texts in Mathematics, 2007 Publication	<1 %
16	Silvia Faggian. "Regular Solutions of First-Order Hamilton–Jacobi Equations for Boundary Control Problems and Applications to Economics", Applied Mathematics and Optimization, 2005 Publication	<1 %
17	journalofinequalitiesandapplications.springeropen.com	

<1 %

18

Adrien Koessler, Alexandre Goldsztejn, Sebastien Briot, Nicolas Bouton. "Dynamics-Based Algorithm for Reliable Assembly Mode Tracking in Parallel Robots", IEEE Transactions on Robotics, 2020

Publication

<1 %

19

L. Bessières. "Collapsing irreducible 3-manifolds with nontrivial fundamental group", Inventiones mathematicae, 11/11/2009

Publication

<1 %

20

Yoshikazu Giga, Tetsuro Miyakawa. " Navier-stokes flow in \mathbb{R}^n with measures as initial vorticity and morrey spaces ", Communications in Partial Differential Equations, 1989

Publication

<1 %

21

ifsc.com

Internet Source

<1 %

22

orcid.org

Internet Source

<1 %

23

www.journaloffinequalitiesandapplications.com

Internet Source

<1 %

24

Arita, M.. "Thermodynamics and defect structures of silver sulfide", Journal of Physics and Chemistry of Solids, 200709

Publication

<1 %

25

Pekka Koskela. "Removable sets for Sobolev spaces", Arkiv för matematik, 10/1999

Publication

<1 %

26

"Stokes–Darcy Equations", 'Springer Science and Business Media LLC', 2019

Internet Source

<1 %

27 Hassen Aydi. "On a Ginzburg-Landau type energy with discontinuous constraint for high values of applied field", Chinese Annals of Mathematics, Series B, 2010 <1 %
Publication

28 Mubariz G. Hajibayov. "Generalized potentials in variable exponent Lebesgue spaces on homogeneous spaces", Mathematische Nachrichten, 01/2011 <1 %
Publication

29 Nurzhan Bokayev, Victor Burenkov, Dauren Matin. "Sufficient conditions for the pre-compactness of sets in global Morrey-type spaces", AIP Publishing, 2017 <1 %
Publication

Exclude quotes On

Exclude matches < 5 words

Exclude bibliography On