Some function spaces and their applications to elliptic partial differential equations

Article · January 2023

DOI: 10.57016/MV-cdyn1783

3 authors:

Nicky Kurnia Tumalun
Universitas Negeri Manado

Denny Ivanal Hakim
Bandung Institute of Technology

Hendra Gunawan
Bandung Institute of Technology

MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 26.01.2023

research paper
оригинални научни рад
DOI: 10.57016/MV-cdyn1783

SOME FUNCTION SPACES AND THEIR APPLICATIONS TO ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Nicky K. Tumalun, Denny I. Hakim and Hendra Gunawan

Abstract. In this paper we prove Fefferman's inequalities associated to potentials belonging to a generalized Morrey space or a Stummel class. We also show that the logarithm of a non-negative weak solution to a second order elliptic partial differential equation with potential in a generalized Morrey space or a Stummel class, under some assumptions, belongs to the bounded mean oscillation class. As a consequence, this elliptic partial differential equation has the strong unique continuation property. An example of an elliptic partial differential equation with potential in a Morrey space or a Stummel class which does not satisfy the strong unique continuation is presented.

1. Introduction and statement of main results

Let $1 \leq p < \infty$ and $\varphi : (0, \infty) \to (0, \infty)$. The **generalized Morrey space** $L^{p,\varphi} := L^{p,\varphi}(\mathbb{R}^n)$, which was introduced by Nakai in [14], is the collection of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{L^{p,\varphi}} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{\varphi(r)} \int\limits_{|x-y|<r} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Note that $L^{p,\varphi}$ is a Banach space with norm $\| \cdot \|_{L^{p,\varphi}}$. If $\varphi(r) = 1$, then $L^{p,\varphi} = L^p$. If $\varphi(r) = r^n$, then $L^{p,\varphi} = L^\infty$. If $\varphi(r) = r^\lambda$ where $0 < \lambda < n$, then $L^{p,\varphi} = L^{p,\lambda}$ is the classical Morrey space introduced in [12].

We will assume the following conditions for φ which will be stated whenever necessary.

(i) There exists $C > 0$ such that

$$s \leq t \Rightarrow \varphi(s) \leq C\varphi(t). \tag{1}$$

We say that φ is **almost increasing** if φ satisfies this condition.

2020 Mathematics Subject Classification: 26D10, 46E30, 35J15
Keywords and phrases: Morrey spaces; Stummel classes; Fefferman's inequality; srong unique continuation property.

(ii) There exists $C > 0$ such that

$$s \leq t \Rightarrow \frac{\varphi(s)}{s^n} \geq C\frac{\varphi(t)}{t^n}. \tag{2}$$

We say that $\varphi(t)t^{-n}$ is **almost decreasing** if $\varphi(t)t^{-n}$ satisfies this condition.

(iii) For $1 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, there exists a constant $C > 0$ such that for every$\delta > 0$,

$$\int_\delta^\infty \frac{\varphi(t)}{t^{(n+1)-\frac{p}{2}(\alpha+1)}}\, dt \leq C\delta^{\frac{p}{2}(1-\alpha)}. \tag{3}$$

One can check that the function $\varphi(t) = t^{n-\alpha p}$, $t > 0$, satisfies all conditions (1), (2), and (3). Moreover, for a non-trivial example, we have the function $\varphi_0(t) = \log(\varphi(t)+1) = \log(t^{n-\alpha p}+1)$, $t > 0$, which satisfies all conditions above.

Let M be the **Hardy-Littlewood maximal operator**, defined by

$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The function $M(f)$ is called the **Hardy-Littlewood maximal function**. Notice that, for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ where $1 \leq p \leq \infty$, $M(f)(x)$ is finite for almost all $x \in \mathbb{R}^n$. Using Lebesgue Differentiation Theorem, we have $|f(x)| \leq M(f)(x)$, for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and for almost all $x \in \mathbb{R}^n$. Furthermore, for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ where $1 \leq p \leq \infty$ and $0 < \gamma < 1$, the nonnegative function $w(x) := [M(f)(x)]^\gamma$ is an A_1 weight, that is, $M(w)(x) \leq C(n,\gamma)w(x)$. These fundamental properties can be found in [6].

We will need the boundedness result for the Hardy-Littlewood maximal operator on generalized Morrey spaces $L^{p,\varphi}$, that is, $\|M(f)\|_{L^{p,\varphi}} \leq C(n,p)\|f\|_{L^{p,\varphi}}$, for every $f \in L^{p,\varphi}$, where $1 \leq p < \infty$ and φ satisfies conditions (1) and (2). This boundedness result was stated in [14, 15, 18]. Our assumptions here on φ are similar to [18]. Note that in [14], the proof of this boundedness result relies on a condition about the integrability of $\varphi(t)t^{-(n+1)}$ over the interval (δ,∞) for every positive number δ. Meanwhile, other assumptions on φ can be found in [15].

Let $1 \leq p < \infty$ and $0 < \alpha < n$. For $V \in L^p_{\text{loc}}(\mathbb{R}^n)$, we write

$$\eta_{\alpha,p}V(r) := \sup_{x\in\mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|V(y)|^p}{|x-y|^{n-\alpha}}\, dy \right)^{\frac{1}{p}}, \quad r > 0.$$

We call $\eta_{\alpha,p}V$ the **Stummel p-modulus** of V. If $\eta_{\alpha,p}V(r)$ is finite for every $r > 0$, then $\eta_{\alpha,p}V(r)$ is nondecreasing on the set of positive real numbers and satisfies $\eta_{\alpha,p}V(2r) \leq C(n,\alpha)\, \eta_{\alpha,p}V(r)$, for every $r > 0$. The last inequality is known as the **doubling condition** for the Stummel p-modulus of V [21, p.550].

For each $0 < \alpha < n$ and $1 \leq p < \infty$, let $\tilde{S}_{\alpha,p} := \{V \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta_{\alpha,p}V(r) < \infty \text{ for all } r > 0\}$ and $S_{\alpha,p} := \{V \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta_{\alpha,p}V(r) < \infty \text{ for all } r > 0 \text{ and } \lim_{r\to 0} \eta_{\alpha,p}V(r) = 0\}$. The set $S_{\alpha,p}$ is called a **Stummel class**, while $\tilde{S}_{\alpha,p}$ is called a **bounded Stummel modulus class**. For $p = 1$, $S_{\alpha,1} := S_\alpha$ are the Stummel classes which were introduced in [17]. We also write $\tilde{S}_{\alpha,1} := \tilde{S}_\alpha$ and $\eta_{\alpha,1} := \eta_\alpha$. It was shown in [21] that $\tilde{S}_{\alpha,p}$ contains $S_{\alpha,p}$ properly. These classes play an important role in studying the regularity theory of partial differential equations (see [23] for example), and have an inclusion relation with Morrey spaces under certain conditions [20, 21].

Now we state our results for Fefferman's inequalities.

THEOREM 1.1. *Let $1 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and φ satisfy conditions* (1), (2), (3). *If $V \in L^{p,\varphi}$, then*

$$\int_{\mathbb{R}^n} |u(x)|^\alpha |V(x)|\, dx \le C\|V\|_{L^{p,\varphi}} \int_{\mathbb{R}^n} |\nabla u(x)|^\alpha\, dx \tag{4}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$.

THEOREM 1.2. *Let $1 \le p < \infty$, $1 \le \alpha \le 2$, and $\alpha < n$. If $V \in \tilde{S}_{\alpha,p}(\mathbb{R}^n)$, then there exists a constant $C := C(n,\alpha) > 0$ such that*

$$\int_{B(x_0,r_0)} |V(x)|^p |u(x)|^\alpha\, dx \le C[\eta_{\alpha,p}V(r_0)]^p \int_{B(x_0,r_0)} |\nabla u(x)|^\alpha\, dx,$$

for every ball $B_0 := B(x_0, r_0) \subseteq \mathbb{R}^n$ and $u \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(u) \subseteq B_0$.

REMARK 1.3. The assumption that the function u belongs to $C_0^\infty(\mathbb{R}^n)$ in Theorem 1.1 and Theorem 1.2 can be weakened by the assumption that u has a weak gradient in a ball $B \subset \mathbb{R}^n$ and a compact support in B (see [22, p.480]).

In 1983, C. Fefferman [5] proved Theorem 1.1 for the case $V \in L^{p,n-2p}$, where $1 < p \le \frac{n}{2}$. The inequality (4) is now known as **Fefferman's inequality**. Chiarenza and Frasca [2] extended the result [5] by proving Theorem 1.1 under the assumption that $V \in L^{p,n-\alpha p}$, where $1 < \alpha < n$ and $1 < p \le \frac{n}{\alpha}$. By setting $\varphi(t) = t^{n-\alpha p}$ in Theorem 1.1, we can recover the results in [2] and [5]. There is also an inequality stated in [19, Proposition 1.8] which may be related to Theorem 1.1. However we cannot compare this inequality with Theorem 1.1.

For the particular case where $V \in \tilde{S}_2$, Theorem 1.2 was proved by Zamboni [23], and can be also concluded by applying the result Fabes *et al.* in [4, p.197] with an additional assumption that V is a radial function. Although $\tilde{S}_\alpha \subset \tilde{S}_2$ whenever $1 \le \alpha \le 2$ [21, p.553], the authors still do not know how to deduce Theorem 1.2 from this result.

It must be noted that Theorems 1.1 and 1.2 are independent to each other, which means that $L^{p,n-\alpha p}$, where $1 < \alpha < n$ and $1 < p \le \frac{n}{\alpha}$, is not contained in $S_{\alpha,p}$. Conversely, $S_{\alpha,p}$ is not contained in $L^{p,n-\alpha p}$. Indeed, if we define $V_1 : \mathbb{R}^n \to \mathbb{R}$ by the formula $V_1(y) := |y|^{-\alpha}$, then $V_1 \in L^{p,n-\alpha p}$, but $V_1 \notin \tilde{S}_{\alpha,p}$. For the function $V_2 : \mathbb{R}^n \to \mathbb{R}$ which is defined by the formula $V_2(y) := |y|^{-\frac{1}{p}}$, we have $V_2 \in \tilde{S}_{\alpha,p}$, but $V_2 \notin L^{p,n-\alpha p}$.

In order to apply Theorems 1.1 and 1.2, let us recall the following definitions. Let Ω be an open and bounded subset of $\mathbb{R}^n$. Recall that the **Sobolev space** $H^1(\Omega)$ is the set of all functions $u \in L^2(\Omega)$ for which the weak derivative $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$

for all $i = 1, \ldots, n$, and is equipped by the **Sobolev norm** $\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}$. The closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ under the Sobolev norm is denoted by $H_0^1(\Omega)$.

Define the operator L on $H_0^1(\Omega)$ by

$$Lu := -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + Vu \tag{5}$$

where $a_{ij} \in L^\infty(\Omega)$, b_i $(i, j = 1, \ldots, n)$ and V is a real valued measurable function on $\mathbb{R}^n$. Throughout this paper, we assume that the matrix $a(x) := (a_{ij}(x))$ is symmetric on Ω and that the ellipticity and boundedness conditions

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \tag{6}$$

hold for some $\lambda > 0$, for all $\xi \in \mathbb{R}^n$, and for almost all $x \in \Omega$.

In (5), we assume either:

$$\begin{cases} \varphi \text{ satisfies } (1), (2), (3)\ (1 < \alpha \leq 2), \\ b_i^2 \in L^{p,\varphi},\ i = 1, \ldots, n, \\ V \in L^{p,\varphi} \cap L^2_{\text{loc}}(\mathbb{R}^n), \end{cases} \tag{7}$$

or,

$$\begin{cases} 1 \leq \alpha \leq 2, \\ b_i^2 \in \tilde{S}_\alpha,\ i = 1, \ldots, n, \\ V \in \tilde{S}_\alpha. \end{cases} \tag{8}$$

We say that $u \in H_0^1(\Omega)$ is a **weak solution** to the equation

$$Lu = 0 \tag{9}$$

if

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i}\psi + Vu\psi \right) dx = 0, \tag{10}$$

for all $\psi \in H_0^1(\Omega)$ (see the definition in [23]). Note that, in the case $\alpha = 2$, the equation (9) was considered in [23]. If we choose $b_i = 0$ for all $i = 1, \ldots, n$, then (9) becomes the Schrödinger equation.

A locally integrable function f on $\mathbb{R}^n$ is said to be of **bounded mean oscillation** on a ball $B \subseteq \mathbb{R}^n$, we write $f \in BMO_\alpha(B)$ where $1 \leq \alpha < \infty$, if there is a constant $C > 0$ such that for every ball $B' \subseteq B$, $\left(\frac{1}{|B'|}\int_{B'} |f(y) - f_{B'}|^\alpha\, dy\right)^{\frac{1}{\alpha}} \leq C$. By using Hölder's inequality and the John-Nirenberg theorem (see [16]), we can prove that $BMO_\alpha(B) = BMO_1(B) := BMO(B)$.

As an application of Theorems 1.1 and 1.2 to equation $Lu = 0$ (9), we have the following result.

Theorem 1.4. *Suppose that α satisfies (7) or (8). Let $u \geq 0$ be the weak solution to the equation $Lu = 0$ and $B(x, 2r) \subseteq \Omega$ where $r \leq 1$. Then* $\log(u + \delta) \in$

$BMO_\alpha(B(x,r))$ for every $\delta > 0$.

In the case $\alpha = 2$, Theorem 1.4 was obtained in [23]. To the best of our knowledge, the assumptions in (7) have never been used for proving Theorem 1.4 as well as the assumption $\alpha \in [1, 2)$ in (8).

Let $w \in L^1_{\text{loc}}(\Omega)$ and $w \geq 0$ in Ω. The function w is said to **vanish with infinite order** at $x_0 \in \Omega$ if $\lim_{r\to 0} \frac{1}{|B(x_0,r)|^k} \int_{B(x_0,r)} w(x)\,dx = 0$, $\forall k > 0$. The equation $Lu = 0$, which is given in (9), is said to have the **strong unique continuation property** in Ω if for every nonnegative solution u which vanishes with infinite order at some $x_0 \in \Omega$, then $u \equiv 0$ in $B(x_0, r)$ for some $r > 0$. See this definition, for example in [7, 10].

Theorem 1.4 gives the following result.

Corollary 1.5. *The equation $Lu = 0$ has the strong unique continuation property in Ω.*

This strong unique continuation property was studied by several authors. For example, Chiarenza and Garofalo in [2] discussed the Schrödinqer inequality of the form $Lu + Vu \geq 0$, where the potential V belongs to Lorentz spaces $L^{\frac{n}{2},\infty}(\Omega)$. For the differential inequality of the form $|\Delta u| \leq |V||u|$, where its potential also belong to $L^{\frac{n}{2}}(\Omega)$, see Jerison and Kenig [10]. Garofalo and Lin [7] studied the equation (9) where the potentials are bounded by certain functions.

Fabes et al. studied the strong unique continuation property for Schrödinqer equation $-\Delta u + Vu = 0$, where the assumption for V is radial function in S_2 [4]. Meanwhile, Zamboni [23] also studied the equation (9) under the assumption that the potentials belong to S_2. At the end of this paper, we will give an example of Schrödinqer equation $-\Delta u+Vu = 0$ that does not satisfy the strong unique continuation property, where $V \in L^{p,n-4p}$ or $V \in \tilde{S}_\beta$ for all $\beta \geq 4$.

2. Proofs

In this section, we prove Fefferman's inequalities, which have been state as Theorems 1.1 and 1.2 above. First, we start with the case where the potential belongs to a generalized Morrey space. Second, we consider the potential from a Stummel class. Furthermore, we present an inequality which is deduced from this inequality.

2.1 Fefferman's Inequality in Generalized Morrey Spaces

We start with the following lemma for potentials in generalized Morrey spaces.

Lemma 2.1. *Let $1 < p < \infty$ and φ satisfy the conditions* (1) *and* (2). *If $1 < \gamma < p$ and $V \in L^{p,\varphi}$, then $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in A_1 \cap L^{p,\varphi}$.*

The proof of Lemma 2.1 uses a fundamental fact from [6] and the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces.

Lemma 2.2. *Let φ satisfy the conditions* (1), (2), *and* (3). *If $V \in L^{p,\varphi}$, then*

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \le C(n,\alpha,p)\|V\|_{L^{p,\varphi}}^{\frac{1}{\alpha}}[M(V)(x)]^{\frac{\alpha-1}{\alpha}}.$$

Proof. Let $\delta > 0$. Then

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy = \int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy + \int_{|x-y|\ge\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy. \tag{11}$$

Using [9, Lemma (a)], we have

$$\int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \le C(n)M(V)(x)\delta. \tag{12}$$

For the second term on the right-hand side (11), let $q = n - \frac{p}{2}(\alpha+1)$, we use Hölder's inequality to obtain

$$\begin{aligned}\int_{|x-y|\ge\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy &= \int_{|x-y|\ge\delta} \frac{|V(y)||x-y|^{\frac{q}{p}+1-n}}{|x-y|^{\frac{q}{p}}}\,dy\\ &\le \left(\int_{|x-y|\ge\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy\right)^{\frac{1}{p}} \times \left(\int_{|x-y|\ge\delta} |x-y|^{(\frac{q}{p}+1-n)(\frac{p}{p-1})}\,dy\right)^{\frac{p-1}{p}}.\end{aligned} \tag{13}$$

By applying the condition (3), we have

$$\begin{aligned}\int_{|x-y|\ge\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy &= \sum_{k=0}^{\infty} \int_{2^k\delta\le|x-y|<2^{k+1}\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy\\ &\le C\|V\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} \frac{\varphi(2^{k+1}\delta)}{(2^k\delta)^{q+1}} \int_{2^{k+1}\delta}^{2^{k+2}\delta} 1dt \le C\|V\|_{L^{p,\varphi}}^p \int_{\delta}^{\infty} \frac{\varphi(t)}{t^{q+1}}\,dt\\ &\le C\|V\|_{L^{p,\varphi}}^p \delta^{n-p\alpha-q}.\end{aligned} \tag{14}$$

Since $n + (\frac{q}{p}+1-n)(\frac{p}{p-1}) < 0$, we obtain

$$\int_{|x-y|\ge\delta} |x-y|^{(\frac{q}{p}+1-n)(\frac{p}{p-1})}\,dy = C(n,p,\alpha)\delta^{n+(\frac{q}{p}+1-n)(\frac{p}{p-1})}. \tag{15}$$

Introducing (14) and (15) in (13), we have

$$\begin{aligned}\int_{|x-y|\ge\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy &\le C\|V\|_{L^{p,\varphi}} \left(\delta^{n-p\alpha-q}\right)^{\frac{1}{p}} \left(\delta^{n+(\frac{q}{p}+1-n)(\frac{p}{p-1})}\right)^{\frac{p-1}{p}}\\ &= C\|V\|_{L^{p,\varphi}}\delta^{1-\alpha}.\end{aligned} \tag{16}$$

From (16), (12) and (11), we get

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \le CM(V)(x)\delta + C\|V\|_{L^{p,\varphi}}\delta^{1-\alpha} \tag{17}$$

For $\delta = \|V\|_{L^{p,\varphi}}^{\frac{1}{\alpha}}[M(V)(x)]^{-\frac{1}{\alpha}}$, the inequality (17) becomes

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \le C[M(V)(x)]^{1-\frac{1}{\alpha}}\|V\|_{L^{p,\varphi}}^{\frac{1}{\alpha}} = C[M(V)(x)]^{\frac{\alpha-1}{\alpha}}\|V\|_{L^{p,\varphi}}^{\frac{1}{\alpha}}. \qquad \square$$

Now, we are ready to prove Fefferman's inequality in generalized Morrey spaces.

Proof (of Theorem 1.1). Let $1 < \gamma < p$ and $w := [M(|V|^\gamma)]^{\frac{1}{\gamma}}$. Then $w \in A_1 \cap L^{p,\varphi}$ according to Lemma 2.1. First, we will show that (4) holds for w in place of V. For any $u \in C_0^\infty(\mathbb{R}^n)$, let B be a ball such that $u \in C_0^\infty(B)$. From the well-known inequality

$$|u(x)| \le C \int_{B_0} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy, \tag{18}$$

Tonelli's theorem, and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x)|^\alpha w(x)\, dx &= \int_B |u(x)|^\alpha w(x)\, dx \\ &\le C\|w\|_{L^{p,\varphi}}^{\frac{1}{\alpha}} \int_B |u(x)|^{\alpha-1} |\nabla u(x)| [M(w)(x)]^{\frac{\alpha-1}{\alpha}}\, dx. \end{aligned}\tag{19}$$

Hölder's inequality and Lemma (2.1) imply that

$$\begin{aligned}\int_B |u(x)|^{\alpha-1}|\nabla u(x)|[M(w)(x)]^{\frac{\alpha-1}{\alpha}}\, dx &\le \left(\int_B |\nabla u(x)|^\alpha\, dx\right)^{\frac{1}{\alpha}} \left(\int_B |u(x)|^\alpha M(w)(x)\, dx\right)^{\frac{\alpha-1}{\alpha}} \\ &\le C\left(\int_B |\nabla u(x)|^\alpha\, dx\right)^{\frac{1}{\alpha}} \left(\int_B |u(x)|^\alpha w(x)\, dx\right)^{\frac{\alpha-1}{\alpha}}. \end{aligned}\tag{20}$$

Substituting (20) into (19), we obtain

$$\int_{\mathbb{R}^n} |u(x)|^\alpha |w(x)|\, dx \le C\|w\|_{L^{p,\varphi}}^{\frac{1}{\alpha}} \left(\int_B |\nabla u(x)|^\alpha\, dx\right)^{\frac{1}{\alpha}} \left(\int_B |u(x)|^\alpha w(x)\, dx\right)^{\frac{\alpha-1}{\alpha}}.$$

Therefore, $\int_{\mathbb{R}^n} |u(x)|^\alpha w(x)\, dx \le C\|w\|_{L^{p,\varphi}} \int_B |\nabla u(x)|^\alpha\, dx$ and $|V(x)| = [|V(x)|^\gamma]^{\frac{1}{\gamma}} \le [M(|V(x)|^\gamma)]^{\frac{1}{\gamma}} = w(x)$. Hence, from the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces and Lemma 2.1, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x)|^\alpha |V(x)|\, dx &\le \int_{\mathbb{R}^n} |u(x)|^\alpha w(x)\, dx \le C\|w\|_{L^{p,\varphi}} \int_B |\nabla u(x)|^\alpha\, dx \\ &\le C\|V\|_{L^{p,\varphi}} \int_{\mathbb{R}^n} |\nabla u(x)|^\alpha\, dx. \end{aligned}$$

□

We have already shown in Theorem 1.1 that Fefferman's inequality holds in generalized Morrey spaces under certain conditions.

2.2 Fefferman's Inequality in Stummel Classes

We need the following lemma to prove Fefferman's inequality where its potentials belong to Stummel classes. This lemma can be proved by Hedberg's trick [9]. For the case $\alpha = 2$, this lemma can also be deduced from the property of the Riesz kernel which is stated in [11, p. 45].

Lemma 2.3. *Let $1 < \alpha \le 2$ and $\alpha < n$. For any ball $B_0 \subset \mathbb{R}^n$, the following inequality holds:*

$$\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy \le \frac{C}{|x-z|^{\frac{n-1}{\alpha-1}-1}}, \quad x, z \in B_0, \quad x \ne z.$$

The following theorem is Fefferman's inequality where the potential belongs to a Stummel class.

Proof (of Theorem 1.2). The proof is separated into two cases, namely $\alpha = 1$ and $1 < \alpha \le 2$. We first consider the case $\alpha = 1$. Using the inequality (18) together with Fubini's theorem, we get

$$\begin{aligned}\int_{B_0} |u(x)||V(x)|^p\,dx &\le C\int_{B_0} |\nabla u(y)| \int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}}\,dx\,dy\\ &\le C\int_{B_0} |\nabla u(y)| \int_{B(y,2r_0)} \frac{|V(x)|^p}{|x-y|^{n-1}}\,dx\,dy.\end{aligned}$$

It follows from the last inequality and the doubling property of Stummel p-modulus of V that $\int_{B_0} |u(x)||V(x)|^p\,dx \le C\,\eta_{\alpha,p}V(r_0)\int_{B_0} |\nabla u(x)|\,dx$, as desired.

We now consider the case $1 < \alpha \le 2$. Using the inequality (18) and Hölder's inequality, we have

$$\begin{aligned}\int_{B_0} |u(x)|^\alpha |V(x)|^p\,dx &\le C\int_{B_0} |\nabla u(y)| \int_{B_0} \frac{|u(x)|^{\alpha-1}|V(x)|^p}{|x-y|^{n-1}}\,dx\,dy\\ &\le C\left(\int_{B_0} |\nabla u(y)|^\alpha\right)^{\frac{1}{\alpha}} \left(\int_{B_0} F(y)^{\frac{\alpha}{\alpha-1}}\,dy\right)^{\frac{\alpha-1}{\alpha}}, \qquad (21)\end{aligned}$$

where $F(y) := \displaystyle\int_{B_0} \frac{|u(x)|^{\alpha-1}|V(x)|^p}{|x-y|^{n-1}}\,dx,\ y \in B_0$. Applying Hölder's inequality again, we have

$$F(y) \le \left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}}\,dx\right)^{\frac{1}{\alpha}} \left(\int_{B_0} \frac{|u(z)|^\alpha |V(z)|^p}{|z-y|^{n-1}}\,dz\right)^{\frac{\alpha-1}{\alpha}},$$

so that

$$\begin{aligned}\int_{B_0} F(y)^{\frac{\alpha}{\alpha-1}}\,dy &\le \int_{B_0} \left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}}\,dx\right)^{\frac{1}{\alpha-1}} \int_{B_0} \frac{|u(z)|^\alpha |V(z)|^p}{|z-y|^{n-1}}\,dz\,dy\\ &= \int_{B_0} |u(z)|^\alpha |V(z)|^p G(z)\,dz, \qquad (22)\end{aligned}$$

where $G(z) := \displaystyle\int_{B_0}\left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}|z-y|^{(n-1)(\alpha-1)}}\,dx\right)^{\frac{1}{\alpha-1}} dy,\ z \in B_0$. By virtue of Minkowski's integral inequality (or Fubini's theorem for $\alpha = 2$), we see that

$$G(z)^{\alpha-1} \le \int_{B_0} |V(x)|^p \left(\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy\right)^{\alpha-1} dx. \qquad (23)$$

Combining (23), doubling property of Stummel p-modulus of V, and the inequality in Lemma 2.1, we obtain

$$G(z) \le C\left(\int_{B_0} \frac{|V(x)|^p}{|x-z|^{n-\alpha}}\,dx\right)^{\frac{1}{\alpha-1}} \le C[\eta_{\alpha,p}V(r_0)]^{\frac{p}{\alpha-1}}. \qquad (24)$$

Now, (22) and (24) give

$$\int_{B_0} |F(y)|^{\frac{\alpha}{\alpha-1}}\, dy \le C[\eta_{\alpha,p}V(r_0)]^{\frac{p}{\alpha-1}} \int_{B_0} |u(x)|^{\alpha}|V(x)|^p\, dx. \tag{25}$$

Therefore, from (21) and (25), we get

$$\int_{B_0} |u(x)|^{\alpha}|V(x)|^p\, dx$$
$$\le C[\eta_{\alpha,p}V(r_0)]^{\frac{p}{\alpha}} \left(\int_{B_0} |\nabla u(x)|^{\alpha}\, dx\right)^{\frac{1}{\alpha}} \left(\int_{B_0} |u(x)|^{\alpha}|V(x)|^p\, dx\right)^{\frac{\alpha-1}{\alpha}}. \tag{26}$$

Dividing both sides by the third term of the right-hand side of (26), we get the desired inequality. □

3. Applications to elliptic partial differential equations

The two lemmas below tell us that if a function vanishes with infinite order at some $x_0 \in \Omega$ and fulfills the doubling integrability over some neighborhood of x_0, then the function must be identically zero in the neighborhood.

Lemma 3.1 ([8]). *Let $w \in L^1_{\mathrm{loc}}(\Omega)$ and $B(x_0, r) \subseteq \Omega$. Assume that there exists a constant $C > 0$ satisfying $\int_{B(x_0,r)} w(x)\, dx \le C \int_{B\left(x_0,\frac{r}{2}\right)} w(x)\, dx$. If w vanishes with infinite order at x_0, then $w \equiv 0$ in $B(x_0, r)$.*

Lemma 3.2. *Let $w \in L^1_{\mathrm{loc}}(\Omega)$ and $B(x_0, r) \subseteq \Omega$, and $0 < \beta < 1$. Assume that there exists a constant $C > 0$ satisfying $\int_{B(x_0,r)} w^{\beta}(x)\, dx \le C \int_{B\left(x_0,\frac{r}{2}\right)} w^{\beta}(x)\, dx$. If w vanishes with infinite order at x_0, then $w \equiv 0$ in $B(x_0, r)$.*

The proof of Lemma 3.2 can be adapted after that of Lemma 3.1 (see [3,8]).

The following lemma has been used by many authors in working with elliptic partial differential equations (for example, see [3,23]). This lemma and the idea of its proof can be found in [13].

Lemma 3.3. *Let $w : \Omega \to \mathbb{R}$ and $B(x, 2r)$ be an open ball in Ω. If $\log(w) \in BMO(B)$ with $B = B(x, r)$, then there exists $M > 0$ such that $\int_{B(x,2r)} w^{\beta}(y)\, dy \le M^{\frac{1}{2}} \int_{B(x,r)} w^{\beta}(y)\, dy$ for some $0 < \beta \le 1$.*

Theorems 1.1 and 1.2 are crucial in proving Theorem 1.4.

Proof (of Theorem 1.4). Let$\delta > 0$ be given. Since $u \in H^1_0(\Omega)$ and $u \ge 0$, then there is a sequence $\{u_k\}_{k=1}^{\infty}$ in $C_0^{\infty}(\Omega)$ such that $u_k + \delta > 0$, for every $k \in \mathbb{N}$, $u_k + \delta \to u + \delta$ a.e in Ω, and $\lim_{k\to\infty} \|u_k - u\|_{H^1(\Omega)} = 0$ (see [1, p.94]).

Let $\psi \in C_0^{\infty}(B(x, 2r))$, $0 \le \psi \le 1$, $|\nabla\psi| \le C_1 r^{-1}$, and $\psi := 1$ on $B(x, r)$. For every $k \in \mathbb{N}$, we have $\psi^{\alpha+1}/(u_k + \delta) \in H^1_0(\Omega)$. Using this as a test function in the weak solution definition (10), we obtain

$$\int_\Omega \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} = (\alpha+1)\int_\Omega \langle a\nabla u, \nabla\psi\rangle \frac{\psi^{\alpha}}{(u_k+\delta)} + \sum_{i=1}^n \int_\Omega b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} + \int_\Omega Vu\frac{\psi^{\alpha+1}}{(u_k+\delta)}. \tag{27}$$

Since $\mathrm{supp}(\psi)\subseteq B(x,2r)$, the inequality (27) reduces to

$$\int_{B(x,2r)} \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} = (\alpha+1)\int_{B(x,2r)} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^{\alpha}}{(u_k+\delta)} + \sum_{i=1}^n \int_{B(x,2r)} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} + \int_{B(x,2r)} Vu\frac{\psi^{\alpha+1}}{(u_k+\delta)}. \tag{28}$$

We will estimate all three terms on the right-hand side of (28). For the first term, according to (6), we have

$$|\langle a\nabla u, \nabla\psi\rangle| \le \lambda^{-1}|\nabla u||\nabla\psi|. \tag{29}$$

Combining Young's inequality $sv \le \epsilon s^2 + \frac{1}{4\epsilon}v^2$ for every $\epsilon > 0$ $(s,v>0)$ and the inequality (29), we have for every $\epsilon > 0$

$$\begin{aligned}
&(\alpha+1)\int_{B(x,2r)} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^{\alpha}}{(u_k+\delta)}\\
\le\,&\epsilon\lambda^{-1}(\alpha+1)\int_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{2\alpha} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int_{B(x,2r)} |\nabla\psi|^2\\
\le\,&\epsilon\lambda^{-1}(\alpha+1)\int_{B(x,2r)} \frac{|\nabla (u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int_{B(x,2r)} |\nabla\psi|^2.
\end{aligned} \tag{30}$$

To estimate the second term in (28), we use Hölder's inequality, Young's inequality and Theorem 1.1 or Theorem 1.2, to obtain

$$\begin{aligned}
&\int_{B(x,2r)} b_i\frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} \le \left(\int_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{\alpha+1}\right)^{\frac12}\left(\int_{B(x,2r)} b_i^2\psi^{\alpha+1}\right)^{\frac12}\\
\le\,&\frac{\epsilon}{n}\int_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{1}{4n\epsilon}\int_{B(x,2r)} b_i^2\psi^{\alpha}\\
\le\,&\frac{\epsilon}{n}\int_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{1}{4n\epsilon}C_1^i\int_{B(x,2r)} |\nabla\psi|^{\alpha}.
\end{aligned} \tag{31}$$

for every $i = 1,\dots,n$, where the constants C_1^i's depend on $n, \alpha, \|b_i^2\|_{L^{p,\varphi}}$ or $\eta_\alpha b_i^2(r_0)$.

From (31) we have

$$\sum_{i=1}^{n} \int_{B(x,2r)} b_i \frac{\partial u}{\partial x_i} \frac{\psi^{\alpha+1}}{(u_k+\delta)} \leq \epsilon \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2} \psi^{\alpha+1} + \frac{1}{4\epsilon} C_2 \int_{B(x,2r)} |\nabla \psi|^\alpha, \tag{32}$$

where C_2 depends on $\max_i \{C_1^i\}$. The estimate for the last term in (28) is

$$\int_{B(x,2r)} V u \frac{\psi^{\alpha+1}}{(u_k+\delta)} \leq \int_{B(x,2r)} V \frac{(u+\delta)}{(u_k+\delta)} \psi^\alpha. \tag{33}$$

Introducing (30), (32), and (33) in (28), we get

$$\begin{aligned}
&\int_{B(x,2r)} \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} \\
&\leq \epsilon \lambda^{-1}(\alpha+1) \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2} \psi^{\alpha+1} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon} \int_{B(x,2r)} |\nabla \psi|^2 \\
&+ \epsilon \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2} \psi^{\alpha+1} + \frac{1}{4\epsilon} C_2 \int_{B(x,2r)} |\nabla \psi|^\alpha + \int_{B(x,2r)} V \frac{(u+\delta)}{(u_k+\delta)} \psi^\alpha,
\end{aligned} \tag{34}$$

for every $k \in \mathbb{N}$.

Since $(u_k+\delta) \to (u+\delta)$ a.e. in Ω and $u+\delta > 0$, then

$$\frac{1}{(u_k+\delta)} \to \frac{1}{(u+\delta)}, \text{ a.e. in } \Omega. \tag{35}$$

For $j, i = 1, \ldots, n$, we infer from (35)

$$\frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} \to \frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u+\delta)^2}, \text{ a.e. in } B(x,2r). \tag{36}$$

For every $k \in \mathbb{N}$, we have

$$\left| \frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} \right| \leq \left| \frac{\partial(u+\delta)}{\partial x_j} \right| \left| \frac{\partial u}{\partial x_i} \right| \frac{1}{\delta^2} \tag{37}$$

and

$$\int_{B(x,2r)} \left| \frac{\partial(u+\delta)}{\partial x_j} \right| \left| \frac{\partial u}{\partial x_i} \right| \frac{1}{\delta^2} \leq \frac{1}{\delta^2} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)} < \infty, \tag{38}$$

since $u \in H_0^1(\Omega)$. The properties (36), (37), and (38) allow us to use the Lebesgue Dominated Convergence Theorem to obtain

$$\lim_{k\to\infty} \int_{B(x,2r)} \left| \frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} - \frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u+\delta)^2} \right| = 0. \tag{39}$$

By Hölder's inequality, we also have

$$\int_{B(x,2r)} \left| \left(\frac{\partial(u_k+\delta)}{\partial x_j} - \frac{\partial(u+\delta)}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} \right|$$

$$\leq \frac{1}{\delta^2}\left\|\frac{\partial(u_k)}{\partial x_j}-\frac{\partial u}{\partial x_j}\right\|_{L^2(B(x,2r))}\|u\|_{H^1(\Omega)} \leq \frac{1}{\delta^2}\|u_k-u\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)} \tag{40}$$

for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty}\|u_k-u\|_{H^1(\Omega)}=0$, from (40) we get

$$\lim_{k\to\infty}\int\limits_{B(x,2r)}\left|\left(\frac{\partial(u_k+\delta)}{\partial x_j}-\frac{\partial(u+\delta)}{\partial x_j}\right)\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2}\right|=0. \tag{41}$$

Note that

$$\begin{aligned}&\int_{B(x,2r)}\left|a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial(u_k+\delta)}{\partial x_j}\frac{\psi^{\alpha+1}}{(u_k+\delta)^2}-a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial(u+\delta)}{\partial x_j}\frac{\psi^{\alpha+1}}{(u+\delta)^2}\right|\\
&\leq \frac{\|a_{ij}\|_{L^\infty(\Omega)}}{\delta^2}\int\limits_{B(x,2r)}\left|\frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2}-\frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u+\delta)^2}\right|\\
&\quad+\frac{\|a_{ij}\|_{L^\infty(\Omega)}}{\delta^2}\int\limits_{B(x,2r)}\left|\left(\frac{\partial(u_k+\delta)}{\partial x_j}-\frac{\partial(u+\delta)}{\partial x_j}\right)\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2}\right|,\end{aligned} \tag{42}$$

for all $k \in \mathbb{N}$. Combining (39), (41), and (42), we have

$$\begin{aligned}&\lim_{k\to\infty}\int_{B(x,2r)}\langle a\nabla u,\nabla(u_k+\delta)\rangle\frac{\psi^{\alpha+1}}{(u_k+\delta)^2}\\
&=\sum_{i,j=1}^{n}\lim_{k\to\infty}\int_{B(x,2r)}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial(u_k+\delta)}{\partial x_j}\frac{\psi^{\alpha+1}}{(u_k+\delta)^2}=\int_{B(x,2r)}\langle a\nabla u,\nabla(u+\delta)\rangle\frac{\psi^{\alpha+1}}{(u+\delta)^2}.\end{aligned} \tag{43}$$

From (35),

$$\frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1}\to\frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1},\ \text{a.e. in}\, B(x,2r). \tag{44}$$

For every $k \in \mathbb{N}$, we have

$$\frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1}\leq\frac{1}{\delta^2}|\nabla(u+\delta)|^2, \tag{45}$$

and

$$\int\limits_{B(x,2r)}\frac{1}{\delta^2}|\nabla(u+\delta)|^2\leq\frac{1}{\delta^2}\|u\|_{H^1(\Omega)}<\infty, \tag{46}$$

since $u \in H^1_0(\Omega)$. Therefore, by (44), (45), (46), and Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty}\int\limits_{B(x,2r)}\frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1}=\int\limits_{B(x,2r)}\frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1}. \tag{47}$$

We also have

$$V\frac{(u+\delta)}{(u_k+\delta)}\psi^\alpha\to V\frac{(u+\delta)}{(u+\delta)}\psi^\alpha=V\psi^\alpha,\text{a.e. in}\, B(x,2r) \tag{48}$$

because of (35). For every $k \in \mathbb{N}$, we have

$$\left| V \frac{(u+\delta)}{(u_k+\delta)} \psi^\alpha \right| \le \frac{1}{\delta} |V||u+\delta|. \tag{49}$$

If the assumption (7) holds, then

$$\begin{aligned} \int_{B(x,2r)} \frac{1}{\delta}|V||u+\delta| &\le \frac{1}{\delta} \int_{B(x,2r)} |V||u+\delta| \\ &< \frac{1}{\delta} \left(\int_{B(x,2r)} |V|^2 \right)^{\frac{1}{2}} \left(\int_{B(x,2r)} |u+\delta|^2 \right)^{\frac{1}{2}} < \infty, \end{aligned} \tag{50}$$

since $V \in L^2_{\text{loc}}(\mathbb{R})$ and $u \in H^1_0(\Omega)$. On the other hand, if the assumption (8) holds, then $V \in \tilde{S}_\alpha \subset \tilde{S}_1$ by virtue to [21, p.554]. Therefore, using Theorem 1.2 we have

$$\begin{aligned} \int_{B(x,2r)} \frac{1}{\delta}|V||u+\delta| &\le \frac{1}{\delta} \int_{B(x,2r)} |V| + \frac{1}{4\delta} \int_{B(x,2r)} |V||u+\delta|^2 \\ &\le \frac{1}{\delta} \int_{B(x,2r)} |V| + \frac{1}{4\delta} C(n)\eta_2 V(r) \int_{B(x,2r)} |\nabla u|^2 < \infty, \end{aligned} \tag{51}$$

since $u \in H^1_0(\Omega)$. Combining (48), (49), (50) or (51), we can apply the Lebesgue Dominated Convergence Theorem to have

$$\lim_{k\to\infty} \int_{B(x,2r)} V \frac{(u+\delta)}{(u_k+\delta)} \psi^\alpha = \int_{B(x,2r)} V\psi^\alpha. \tag{52}$$

Theorem 1.1 or Theorem 1.2 allow us to get the estimate

$$\int_{B(x,2r)} V\psi^\alpha \le C_3 \int_{B(x,2r)} |\nabla \psi|^\alpha, \tag{53}$$

where the constant C_3 depends on n, α, and $\|V\|_{L^{p,\varphi}}$ or $\eta_\alpha V(r_0)$. Letting $k \to \infty$ in (34) and applying all informations in (43), (47), (52), and (53), we obtain

$$\begin{aligned} &\int_{B(x,2r)} \langle a\nabla u, \nabla(u+\delta)\rangle \frac{\psi^{\alpha+1}}{(u+\delta)^2} \\ &\le \epsilon\lambda^{-1}(\alpha+1) \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon} \int_{B(x,2r)} |\nabla\psi|^2 \\ &+ \epsilon \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{1}{4\epsilon}C_2 \int_{B(x,2r)} |\nabla\psi|^\alpha + C_3 \int_{B(x,2r)} |\nabla\psi|^\alpha. \end{aligned} \tag{54}$$

Notice that, by the ellipticity condition (6),

$$\lambda \int\limits_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \leq \int\limits_{B(x,2r)} \langle a\nabla u, \nabla(u+\delta)\rangle \frac{\psi^{\alpha+1}}{(u+\delta)^2}.$$

Moreover, by choosing $\epsilon := \frac{1}{2}\frac{\lambda^2}{(\alpha+1)+1}$, the inequality (54) is simplified by

$$\int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \leq C_4 \int_{B(x,2r)} |\nabla\psi|^2 + C_5 \int_{B(x,2r)} |\nabla\psi|^\alpha, \tag{55}$$

where the constant C_4 depends on α and λ, while the constant C_5 depends on C_2 and C_3. Therefore, (55) implies

$$\int_{B(x,r)} |\nabla \log(u+\delta)|^2 \leq \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1}$$

$$\leq C_5 \int_{B(x,2r)} |\nabla\psi|^2 + C_6 \int_{B(x,2r)} |\nabla\psi|^\alpha \leq C\left(r^{-2}r^n + r^{-\alpha}r^n\right) = Cr^{-2}r^n.$$

The last constant C depends on C_4 and C_5. From Hölder's inequality,

$$\left(\frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^\alpha\right)^{\frac{2}{\alpha}} \leq \frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^2 \leq Cr^{-2},$$

whence

$$\frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^\alpha \leq Cr^{-\alpha}. \tag{56}$$

By using Poincaré's inequality together with the inequality (56), the theorem is proved. □

By virtue of Theorem 1.4, we have the following corollary.

Corollary 3.4. *Suppose α satisfies* (7) *or* (8). *Let $u \geq 0$ be a weak solution to the equation $Lu = 0$ and $B(x,2r) \subseteq \Omega$ where $r \leq 1$. Then, for every $\delta > 0$, $\log(u+\delta) \in BMO_\alpha(B(x,r))$.*

Gathering Lemma 3.1, Lemma 3.2, Lemma 3.3, and Corollary 3.4, we obtain the unique continuation property of the equation $Lu = 0$ stated in Corollary 1.5.

Proof (of Corollary 1.5). Given $x \in \Omega$, let $B := B(x,r)$ be a ball where $B(x,2r) \subseteq \Omega$ and $r \leq 1$. Let $\{\delta_j\}$ be a sequence of real numbers in $(0,1)$ which converges to 0. From Corollary 3.4, we get $\log(u+\delta_j) \in BMO_\alpha(B)$. Therefore $\log(u+\delta_j) \in BMO(B)$. According to Lemma 3.3, there exists a constant $M > 0$ such that

$$\int_{B(x,2r)} u^\beta(y)\,dy \leq \int_{B(x,2r)} (u(y)+\delta_j)^\beta\,dy \leq M^{\frac{1}{2}} \int_{B(x,r)} (u(y)+\delta_j)^\beta\,dy,$$

for some $0 < \beta \leq 1$. Letting $j \to \infty$ and using Lemma 3.1 or Lemma 3.2, we obtain $u \equiv 0$ in $B(x,2r)$ if u vanishes with infinite order at x. □

The example below shows that there exists an elliptic partial differential equation which does not satisfy the strong unique continuation property where its potential belongs to Morrey spaces $L^{p,n-4p}$ and $\tilde{S}_\beta$ for all $\beta \geq 4$.

Example 3.5. Let $\Omega = B(0,1) \subseteq \mathbb{R}^n$, $w : \Omega \to \mathbb{R}$ and $V : \mathbb{R}^n \to \mathbb{R}$ be defined by the formulae

$$w(x) = \begin{cases} \exp(-|x|^{-1})|x|^{-(n+1)}, & x \in \Omega\backslash\{0\} \\ 1, & x = 0, \end{cases}$$

and
$$V(x) = \begin{cases} 3(n+1)|x|^{-2} - (n+5)|x|^{-3} + |x|^{-4}, & x \in \mathbb{R}^n\backslash\{0\} \\ 0, & x = 0. \end{cases}$$

Note that w vanishes with infinite order at $x = 0$ and is a solution to the Schrödinger equation $-\Delta u + Vu = 0$. We also have $V \in S_\beta \subseteq \tilde{S}_\beta$, for all $\beta \geq 4$, and $V \notin \tilde{S}_\alpha$, for $1 \leq \alpha \leq 2$.

Define $V^* = V\chi_\Omega$. Then $V^* : \mathbb{R}^n \to \mathbb{R}$ and w is a solution to the equation $-\Delta u + V^* u = 0$. For $y \in \mathbb{R}^n$ and $y \neq 0$, we get $|V^*(y)| \leq (4n+9)|y|^{-4}$. Given $x \in \mathbb{R}^n$ and $r > 0$, by the previous inequality, we have

$$\frac{1}{r^{n-4p}} \int_{|x-y|<r} |V^*(y)|^p \, dy \leq \frac{1}{r^{n-4p}} \int_{|x-y|<r} |y|^{-4p} \, dy = C(n,p). \tag{57}$$

According to (57), we conclude that $V^* \in L^{p,n-4p}$. □

Remark 3.6. The equation $Lu = 0$ has the strong unique continuation property if $V, b_i^2 \in \tilde{S}_\alpha$ for $i = 1, \ldots, n$ and $1 \leq \alpha \leq 2$ (see assumption (8)). In view of Example 3.5, there exist $V \in \tilde{S}_\alpha$, $\alpha \geq 4$, and $b_i = 0$ for $i = 1, \ldots, n$ such that the equation $Lu = 0$ does not have the strong unique continuation property. However, the authors still do not know whether $Lu = 0$ has the strong unique continuation property or not if $V, b_i^2 \in \tilde{S}_\alpha$ for $i = 1, \ldots, n$ and $2 < \alpha < 4$.

Remark 3.7. The equation $Lu = 0$ has the strong unique continuation property if $V, b_i^2 \in L^{p,\varphi}$ where (7) holds. If we choose $V \in L^{p,n-4p}$ (i.e. $\alpha = 4$) as in Example 3.5 and $b_i = 0$ for $i = 1, \ldots, n$, then the equation $Lu = 0$ does not have the strong unique continuation property.

Acknowledgement. This research is supported by ITB Research & Innovation Program 2020. The first author also thanks LPDP Indonesia.

References

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York Dordrecht Heidelberg London, 2010.

[2] F. Chiarenza, M. Frasca, *A remark on a paper by C. Fefferman*, Proc. Amer. Math. Soc. **108** (1990), 407–409.

[3] F. Chiarenza, N. Garofalo, *Unique continuation for nonnegative solutions of Schrödinger operators*, Institute for Mathematics and its Applications, Preprint Series No 122, University of Minnesota, 1984.

[4] E. Fabes, N. Garofalo, F. Lin, *A partial answer to a conjecture of B. Simon concerning unique continuation*, J. Funct. Anal. **88(1)** (1990), 194–210.

[5] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.

[6] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topic*, North Holland, Amsterdam New York Oxford, 1985.

[7] N. Garofalo, F. H. Lin, *Unique continuation for elliptic operators: a geometric-variational approach*, Commun. Pure Appl. Math. **40(3)** (1987), 347–366.

[8] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, New Jersey, 1983.

[9] L.I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[10] D. Jerison, C.E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. Math. **121(3)** (1985), 463–488.

[11] N. S. Landkof, *Fondations of Modern Potential Theory*, Springer-Verlag, Berlin Heidelberg New York, 1972.

[12] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[13] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

[14] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.

[15] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. **287(2)** (2014), 339–351.

[16] L. Pick, A. Kufner, O. John, S. Fucik, *Function Spaces*, Vol. 1. 2nd Revised and Extended Version. De Gruyter, Berlin, 2013.

[17] M.A. Ragusa, P. Zamboni, *A potential theoretic inequality*, Czech. Mat. J. **51** (2001), 55–56.

[18] Y. Sawano, *Generalized Morrey spaces for non-doubling measures*, Nonlinear Differ. Equ. Appl. **15** (2008), 413–425.

[19] Y. Sawano, S. Sugano, H. Tanaka, *Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces*, Trans. Amer. Math. Soc. **363(2)** (2011), 6481–6503.

[20] N. K. Tumalun, H. Gunawan, *Morrey spaces are embedded between weak Morrey spaces and Stummel classes*, J. Indones. Math. Soc. **25(3)** (2019), 203–209.

[21] N.K. Tumalun, D.I. Hakim, H. Gunawan, *Inclusion between generalized Stummel classes and other function spaces*, Math. Inequal. Appl. **23(2)** (2020), 547–562.

[22] R. L. Wheeden, A. Zygmund, *Measure and Integral, An Introduction to Real Analysis*, Second Edition. CRC Press, Boca Raton London New York, 2015.

[23] P. Zamboni, *Some function spaces and elliptic partial differential equations*, Le Matematiche **42(1)** (1987), 171–178.

(received 17.10.2020; in revised form 02.04.2022; available online 26.01.2023)

Mathematics Department, Universitas Negeri Manado, Tondano Selatan, Minahasa Regency 95618, Indonesia

E-mail: nickytumalun@unima.ac.id

Analysis and Geometry Group, Bandung Institute of Technology, Jl. Ganesha No. 10, Bandung 40132, Indonesia

E-mail: dhakim@math.itb.ac.id

Analysis and Geometry Group, Bandung Institute of Technology, Jl. Ganesha No. 10, Bandung 40132, Indonesia

E-mail: hgunawan@math.itb.ac.id